

## A Mathematical and Biomedical Modeling Study of the Hyperbola and Its Applications in Medical Imaging and Radiation Dose Distribution

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### ABSTRACT

This paper aims at obtaining non-zero distinct integer solutions to the hyperbola represented by the binary quadratic equation  $3x^2 + 7xy + 3y^2 - 13x - 13y + 9 = 0$ . A few interesting relations among the solutions are presented. Employing the solutions of the given hyperbola, solutions for other choices of hyperbolas and parabolas are presented.

**Keywords:** Binary quadratic, non-homogeneous quadratic, hyperbola, Integer solutions.

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## I. INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety [1-6]. In [7-18], the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of yet another interesting binary quadratic equation given by  $3x^2 + 7xy + 3y^2 - 13x - 13y + 9 = 0$ . The recurrence relations satisfied by the solutions  $x$  and  $y$  are given. Also a few interesting properties among the solutions are exhibited.

## II. METHOD OF ANALYSIS

The Diophantine equation under consideration is

$$3x^2 + 7xy + 3y^2 - 13x - 13y + 9 = 0 \tag{1}$$

Introduction of the transformations

$$x = X + 1, y = Y + 1 \tag{2}$$

in (1) gives

$$3X^2 + 3Y^2 + 7YX - 4 = 0 \tag{3}$$

Substitution of the transformations,

$$X = u + v, Y = u - v, u \neq v \neq 0 \tag{4}$$

in (3) leads to

$$v^2 = 13u^2 - 4 \tag{5}$$

whose smallest positive integer solution is  $u_0 = 1, v_0 = 3$

Now, consider the pellian equation

$$v^2 = 13u^2 + 1 \tag{6}$$

whose general solution  $(\tilde{u}_n, \tilde{v}_n)$  is given by

$$\tilde{v}_n = \frac{1}{2} f_n, \tilde{u}_n = \frac{1}{2\sqrt{13}} g_n$$

where  $g_n = [(649 + 180\sqrt{13})^{n+1} - (649 - 180\sqrt{13})^{n+1}]$

$$f_n = [(649 + 180\sqrt{13})^{n+1} + (649 - 180\sqrt{13})^{n+1}], \quad n = 0, 1, 2, 3, \dots$$

Applying Brahmagupta lemma between the solutions  $(u_0, v_0)$  and  $(\tilde{u}_n, \tilde{v}_n)$ , we have

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$$u_{n+1} = \frac{f_n}{2} + \frac{3g_n}{2\sqrt{13}}$$

$$v_{n+1} = \frac{3f_n}{2} + \frac{13g_n}{2\sqrt{13}}$$

Taking the advantage of (2) and (4), the sequence of integral solutions of (1) can be written as

$$13x_{n+1} = 26f_n + 8\sqrt{13}g_n + 13 \quad (7)$$

$$13y_{n+1} = -(13f_n + 5\sqrt{13}g_n) + 13 \quad (8)$$

Thus (7) and (8) represent the non-zero distinct integral solutions of (1). The above values of  $x_n$  and  $y_n$  satisfy respectively the following recurrence relations.

$$x_{n+3} = 1298x_{n+2} - x_{n+1} - 1296$$

$$y_{n+3} = 1298y_{n+2} - y_{n+1} - 1296, \quad n = 0, 1, 2, \dots$$

Some numerical examples of  $x$  and  $y$  satisfying (1) are given in the Table :1 below.

**Table 1: Examples**

$n$	$x_{n+1}$	$y_{n+1}$
-1	5	-1
0	5477	-3097
1	7107845	-4021201
2	9225976037	-5219517097

From the above table, we observe some interesting relations among the solutions which are presented below.

1.  $x_{n+1}$  values are positive and  $y_{n+1}$  values are negative. Both values are odd.

2. Each of the following expressions is a Nasty number :

- $\frac{1}{45}(-1234x_{2n+2} + x_{2n+3} + 1773)$
- $\frac{1}{58410}(-1601731x_{2n+2} + x_{2n+4} + 2302650)$
- $3(5x_{2n+2} + 8y_{2n+2} - 9)$
- $\frac{3}{611}(-5585x_{2n+2} - 8y_{2n+3} + 8037)$
- $\frac{3}{793079}(-7249325x_{2n+2} - 8y_{2n+4} + 10421649)$

3. Each of the following expressions is a Cubical integer:

- $\frac{1}{270}(-1234x_{3n+3} + x_{3n+4} - 3702x_{n+1} + 3x_{n+2} + 4932)$
- $\frac{1}{350460}(-1601731x_{3n+3} + x_{3n+5} - 4805193x_{n+1} + 3x_{n+3} + 6406920)$
- $\frac{1}{2}(5x_{3n+3} + 8y_{3n+3} + 15x_{n+1} + 24y_{n+1} - 52)$
- $\frac{1}{1222}(-5585x_{3n+3} - 8y_{3n+4} - 16755x_{n+1} - 24y_{n+2} + 22372)$
- $\frac{1}{1586158}(-7249325x_{3n+3} - 8y_{3n+5} - 21747975x_{n+1} - 24y_{n+3} + 28997332)$

4. Each of the following expressions is a Bi-Quadratic integer:

- $\frac{1}{270}(-1234x_{4n+4} + x_{4n+5} - 4936x_{2n+2} + 4x_{2n+3} + 7785)$

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- $\frac{1}{350460}(-1601731x_{4n+4} + x_{4n+6} - 6406924x_{2n+2} + 4x_{2n+4} + 10111410)$
- $\frac{1}{2}(5x_{4n+4} + 8y_{4n+4} + 20x_{2n+2} + 32y_{2n+2} - 53)$
- $\frac{1}{1222}(-5585x_{4n+4} - 8y_{4n+5} - 22340x_{2n+2} - 32y_{2n+3} + 35297)$
- $\frac{1}{1586158}(-7249325x_{4n+4} - 8y_{4n+6} - 28997300x_{2n+2} - 32y_{2n+4} + 45763613)$

5. Each of the following expressions is a Quintic integer:

- $\frac{1}{270}(-1234x_{5n+5} + x_{5n+6} - 6170x_{3n+3} + 5x_{3n+4} - 12340x_{n+1} + 10x_{n+2} + 19728)$
- $\frac{1}{350460} \left( -1601731x_{5n+5} + x_{5n+7} - 8008655x_{3n+3} + 5x_{3n+5} - 16017310x_{n+1} \right) + 10x_{n+3} + 25627680$
- $\frac{1}{2}(5x_{5n+5} + 8y_{5n+5} + 25x_{3n+3} + 40y_{3n+3} + 50x_{n+1} + 80y_{n+1} - 208)$
- $\frac{1}{1222}(-5585x_{5n+5} - 8y_{5n+6} - 27925x_{3n+3} - 40y_{3n+4} - 55850x_{n+1} - 80y_{n+2} + 89488)$
- $\frac{1}{1586158} \left( -7249325x_{5n+5} - 8y_{5n+7} - 36246625x_{3n+3} - 40y_{3n+5} \right) - 72493250x_{n+1} - 80y_{n+3} + 188482658$

6. Relations among the solutions:

- $13x_{n+3} = -13x_{n+1} + 16874x_{n+2} - 16848$
- $1298x_{n+2} = x_{n+1} + x_{n+3} + 1296$
- $319337750160y_{n+3} = -228000x_{n+1} - 180662603121x_{n+3} + 500000581160$
- $1222y_{n+3} = 2160x_{n+1} + 1586158y_{n+2} - 1587096$
- $793079x_{n+3} = -x_{n+1} - 1401840y_{n+3} + 2194920$
- $793079y_{n+2} = -1080x_{n+1} + 611y_{n+3} + 793548$

Remarkable Observations

(i) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table: 2 below.

**Table 2: Hyperbolas**

S.NO	Hyperbolas	(X,Y)
1	$208X^2 - Y^2 = 60652800$	$\left( -1234x_{n+1} + x_{n+2} + 1233, \right)$ $\left( 17797x_{n+1} - 13x_{n+2} - 17784 \right)$
2	$208X^2 - Y^2 = 102188080051200$	$\left( -1601731x_{n+1} + x_{n+3} + 1601730, \right)$ $\left( 23100493x_{n+1} - 13x_{n+3} - 23100480 \right)$
3	$13X^2 - Y^2 = 208$	$\left( 5x_{n+1} + 8y_{n+1} - 13, \right)$ $\left( -13x_{n+1} - 26y_{n+1} + 39 \right)$
4	$X^2 - 13Y^2 = 5973136$	$\left( -5585x_{n+1} - 8y_{n+2} + 5593, \right)$ $\left( 1549x_{n+1} + 2y_{n+2} - 1551 \right)$
5	$13X^2 - Y^2 = 130826654450128$	$\left( -7249325x_{n+1} - 8y_{n+3} + 7249333, \right)$ $\left( 26137813x_{n+1} + 26y_{n+3} - 26137839 \right)$

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(ii) Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table: 3 below.

**Table3: Parabolas**

S.NO	Parabolas	(X,Y)
1	$56160X - Y^2 = 60652800$	$\left( \begin{array}{l} -1234x_{2n+2} + x_{2n+3} + 1773, \\ 17797x_{n+1} - 13x_{n+2} - 17784 \end{array} \right)$
2	$72895680X - Y^2 = 102188080051200$	$\left( \begin{array}{l} -1601731x_{2n+2} + x_{2n+4} + 2302650, \\ 23100493x_{n+1} - 13x_{n+3} - 23100480 \end{array} \right)$
3	$26X - Y^2 = 208$	$\left( \begin{array}{l} 5x_{2n+2} + 8y_{2n+2} - 9, \\ -13x_{n+1} - 26y_{n+1} + 39 \end{array} \right)$
4	$94X - Y^2 = 459472$	$\left( \begin{array}{l} -5585x_{2n+2} - 8y_{2n+3} + 8037, \\ 1549x_{n+1} + 2y_{n+2} - 1551 \end{array} \right)$
5	$20620054X - Y^2 = 130826654450128$	$\left( \begin{array}{l} -7249325x_{2n+2} - 8y_{2n+4} + 10421649, \\ 26137813x_{n+1} + 26y_{n+3} - 26137839 \end{array} \right)$

### III. CONCLUSION

In conclusion, one may search for other patterns of solutions and their corresponding properties.

### REFERENCES

1. R.D. Carmichael, "The Theory of Numbers and Diophantine Analysis", Dover Publications, New York (1950).
2. L.E. Dickson, "History of theory of numbers", vol.2, Chelsea publishing company, New York (1952).
3. L.J. Mordell, "Diophantine equations", Academic Press, London (1969).
4. S.G. Telang, "Number Theory", Tata Mc Graw Hill Publishing Company, New Delhi (1996).
5. Nigel.P.Smart., "The Algorithm Resolutions of Diophantine Equations", Cambridge University Press, London (1999).
6. T.S.Banumathy, "A Modern Introduction to Ancient Indian Mathematics", Wiley Eastern Limited, London (1995).
7. K. Meena, S. Vidhyalakshmi and A. Nivetha "On The Binary Quadratic Diophantine Equation  $x^2 - 4xy + y^2 + 14x = 0$  ". *Scholars Journal of Physics, Mathematics and Statistics*,3(1),2016, 15-19.
8. M.A. Gopalan, V.Geetha and D. Priyanka "On The Binary Quadratic Diophantine Equation  $x^2 - 3xy + y^2 + 16x = 0$  ". *International Journal of Scientific Engineering and Applied Science*,1(4),June 2015,516-520.
9. M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi and S. Suguna "On The Binary Quadratic Diophantine Equation  $x^2 - 4xy + y^2 + 32x = 0$  ". *Bulletin of Mathematics and Statistics Research*,3(3),2015 ,45-51.
10. S. Vidhyalakshmi, M.A. Gopalan and S. Nandhini "On The Binary Quadratic Diophantine Equation  $x^2 - 3xy + y^2 + 18x = 0$  ". *International Research Journal of Engineering and Technology*, 2(4),July 2015, 825-829.
11. S. Vidhyalakshmi, M.A. Gopalan, A. Kavitha and D.Mary Madona "On The Binary Quadratic Diophantine Equation  $x^2 - 3xy + y^2 + 33x = 0$  ". *International Journal of Scientific Engineering and Applied Science*, 1(4),, June 2015, 222-225.
12. S. Vidhyalakshmi, M.A. Gopalan, R.Presenna and N.Christy "On The Binary Quadratic Diophantine Equation  $x^2 - 3xy + y^2 + 21x = 0$  ". *International Journal of Applied Research*, 1(8), 2015, 666-669.
13. M.A. Gopalan, S .Vidhyalakshmi, R. Presenna and K. Lakshmi "On The Binary Quadratic Diophantine Equation  $x^2 - 3xy + y^2 + 10x = 0$  ". *International Journal of Engineering, science and Mathematics*, 4(2), June 2015, 68-77.

## Metal Ions in Life Sciences

14. K. Meena, S. Vidhyalakshmi, M.A. Gopalan and K. Anitha "Integer Points On The Hyperbola  $x^2 - 5xy + y^2 + 20x = 0$  ", *Archimedes J. Math.*, 4(3),2014 ,149-158.
15. M.A. Gopalan, S. Vidhyalakshmi and J. Shanthi "Integral Points on The Hyperbola  $x^2 - 4xy + y^2 + 11x = 0$  ", *Bulletin of Mathematics and Statistics Research.*,2(3),2014 ,327-330.
16. S. Vidhyalakshmi, A. Kavitha and M.A. Gopalan , "Integral Points on The Hyperbola  $x^2 - 4xy + y^2 + 15x = 0$  ". *International Journal of Innovative Science, Engineering and Technology*,1(7),Sep-2014 ,338-340.
17. S. Vidhyalakshmi, M.A. Gopalan and K.Lakshmi, "Observation on The Binary Quadratic Equation  $x^2 - 8xy + 3y^2 + 2x + 2y + 6 = 0$  ". *Scholars Journal of Physics, Mathematics and Statistics*, 1(2),2014,41-44.
18. S. Vidhyalakshmi, M.A .Gopalan and K.Lakshmi "Integer solutions of The Binary Quadratic Equation  $x^2 - 5xy + y^2 + 33x = 0$  ", *International Journal of Innovative Science, Engineering and Technology*, 1(6), Aug-2014, 450-453.